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On the rate of convergence in singular perturbations of obstacle problems

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§0 Introduction

In this paper we present some results on singular perturbation of obstacle problems of variational inequalities. We investigate the following singular perturbation problems;

$$(1)_\varepsilon \qquad \begin{aligned} \max\{L^\varepsilon u^\varepsilon - f,\ u^\varepsilon - \psi\} &= 0 \quad \text{in } \Omega, \\ u^\varepsilon &= \phi \quad \text{on } \Gamma, \end{aligned}$$

where $\varepsilon > 0$ is a small parameter, $\Omega \subset R^n$ is a bounded domain with smooth boundary Γ and L^ε is a second order elliptic operator of the form

$$L^\varepsilon u = -\varepsilon^2 a_{ij} u_{ij} + \varepsilon b_i u_i + cu.$$

Here and later we use the usual summation convention and the notation: $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$.
We note that the above problem is derived from a variational inequality. For the details of its derivation and motivation we refer [1] and [12].

The work of J.L.Lions [10] is important to the considerations in singular perturbations of obstacle problems. However, his results were obtained in a Hilbert space setting. Therefore, we have not known any results on the rate of pointwise convergence estimates for solutions of obstacle problems. The purpose here is to get the optimal rate of pointwise convergence in singular perturbations of obstacle problems. Our method is due to comparison principle for viscosity solutions.

It is convenient to formulate the notion of viscosity solutions of $(1)_\varepsilon$ in the following manner.

Definition We call a function $u \in C(\overline{\Omega})$ a viscosity solution of $(1)_\varepsilon$ if the following three conditions hold;

(D.1) $u = \phi$ on Γ and $u \leqq \psi$ in Ω.

(D.2) If whenever $\zeta \in C^2(\Omega)$ and $u - \zeta$ attains its local maximum at $x \in \Omega$, then

$$\max\{ -\varepsilon^2 a_{ij}\zeta_{ij} + \varepsilon b_i \zeta_i + cu - f,\ u - \psi\} \leqq 0 \quad \text{at } x.$$

(D.3) If whenever $\zeta \in C^2(\Omega)$ and $u - \zeta$ attains its local minimum at $x \in \Omega$, then

$$\max\{ -\varepsilon^2 a_{ij}\zeta_{ij} + \varepsilon b_i \zeta_i + cu - f,\ u - \psi\} \geqq 0 \quad \text{at } x.$$

In section 1 we shall deal with the problem $(1)_\varepsilon$ under some assumptions, which make a boundary layer phenomenon disappear. In section 2 we shall investigate the problem $(1)_\varepsilon$ under another assumption, which makes a boundary layer of large deviation type arise.

§1 Singular perturbation without boundary layer

In this section we study the following singular perturbations of the obstacle problem:

$$(2)_\varepsilon \quad \begin{aligned} \max\{-\varepsilon^2 a_{ij} u^\varepsilon_{ij} + \varepsilon b_i u^\varepsilon_i + c u^\varepsilon - f,\ u^\varepsilon - \psi\} = 0 \text{ a.e. in } \Omega \\ u^\varepsilon = 0 \quad \text{on } \Gamma, \end{aligned}$$

where a_{ij}, b_i, c, f and ψ are given functions for $i,j = 1,\ldots,n$. For simplicity we assume

(A.1) $\quad a_{ij}, b_i, c, f, \psi \in C^2(\overline{\Omega})$ for $i, j = 1,2,\ldots,n$

and that there is a positive number θ such that

(A.2) $\quad a_{ij}(x)\xi_i\xi_j \geqq \theta|\xi|^2$

for $x \in \overline{\Omega}$, $\xi = (\xi_1,\ldots,\xi_n) \in R^n$.

The boundary condition in $(2)_\varepsilon$ yields the compatibility condition;

(A.3) $\psi \geqq 0$ on Γ.

In order to use comparison principle we also assume that there is a positive number τ such that

(A.4) $c \geqq \tau$ in $\overline{\Omega}$.

Now we shall state our first result:

Theorem 1 Assume hypotheses (A.1)-(A.4). For each $\varepsilon > 0$ let u^ε be the unique solution of $(2)_\varepsilon$. If we assume $\min\{f/c, \psi\} = 0$ on Γ, then we find that there exists a positive constant C such that

$$(3)_\varepsilon \qquad |u^\varepsilon(x) - \min\{f/c, \psi\}(x)| \leqq C\varepsilon$$

for all $x \in \overline{\Omega}$ and small $\varepsilon > 0$.

Remark(1) It is known that for each $\varepsilon > 0$ $(2)_\varepsilon$ has the unique solution in $W^{2,\infty}(\Omega)$; see e.g. [7].

(2) J.L.Lions proved that $\| u^\varepsilon - \min\{f/c, \psi\} \|_{L^2(\Omega)} \leqq C\varepsilon$ under the same assumptions; see p.124 in [11]. Moreover, remarking Proposition 2.3 in [6], we obtain that $u^\varepsilon \to \min\{f/c, \psi\}$ uniformly on $\overline{\Omega}$ as $\varepsilon \downarrow 0$.

(3) Recently N.Yamada [13] has proved that u^ε is the unique viscosity solution of $(2)_\varepsilon$; see also [3] and [5].

<u>Proof of Theorem 1</u> Without loss of generality we may suppose that $c = 1$ in $\overline{\Omega}$. Put $u^0 = \min\{f, \psi\}$.

At first we shall obtain the one-sided inequality: $u^\varepsilon(x) - u^0(x) \leqq C\varepsilon$. Define $\Phi : \overline{\Omega} \times \overline{\Omega} \to R$ by $\Phi(x,y) = u^\varepsilon(x) - u^0(y) - |x - y|^2/\varepsilon - \mu\varepsilon$ for $x, y \in \overline{\Omega}$, where $\mu > \max\{\|Df\|_\infty, \|D\psi\|_\infty\}$ is a positive number to be fixed later on. Assume that Φ attains its positive maximum over $\overline{\Omega} \times \overline{\Omega}$ at a point (x_0, y_0). Using $\Phi(x_0,y_0) \geqq \Phi(x_0,x_0)$, we have

$$(4) \qquad |x_0 - y_0| \leqq C\varepsilon.$$

Here and in the sequel C stands for a various positive constant independent of ε.

If we suppose that $x_0 \in \Gamma$, then we obtain that $\Phi(x_0,y_0) \leqq 0$ for sufficiently large μ. Indeed, (A.1), (4) and the assumption that $u^0(x_0) = 0$ imply $\Phi(x_0,y_0) \leqq (C_0 - \mu)\varepsilon$, where $C_0 = \max\{\|Df\|_\infty, \|D\psi\|_\infty\}$.

Hence, since u^ε is a viscosity solution of $(2)_\varepsilon$, we get

$$0 \geqq \max\{-\varepsilon^2 a_{ij}\zeta_{ij} + \varepsilon b_i \zeta_i + u^\varepsilon - f,\ u^\varepsilon - \psi\}$$

at x_0. Here we take $\zeta(x) = u_0(y_0) + |x - y_0|^2/\varepsilon + \mu\varepsilon$ in (D.2). Using $\zeta_i(x_0) = 2(x_0 - y_0)_i/\varepsilon$ and $\zeta_{ij}(x_0) = 2\delta_{ij}/\varepsilon$, we have

$$0 \geqq u^{\varepsilon} + \max\{-2\varepsilon a_{ii} + 2b_i(x-y_0)_i - f,\ -\psi\} \text{ at } x_0.$$

Remembering $\Phi(x_0, y_0) > 0$ and (A.1), we get

$$0 > u_0(x_0) + \mu\varepsilon - C\varepsilon - u_0(y_0).$$

This implies contradiction for sufficiently large μ.

Therefore, Φ can not attain its positive maximum over $\overline{\Omega} \times \overline{\Omega}$.

Hence we obtain that for all $x \in \overline{\Omega}$

$$u^{\varepsilon}(x) - u^{0}(x) \leqq \sup \Phi + \mu\varepsilon \leqq \mu\varepsilon.$$

If we proceed as above with $\Phi(x,y) = u^0(x) - u^{\varepsilon}(y) - \mu\varepsilon - |x-y|^2/\varepsilon$, then we obtain the converse inequality. Indeed, remarking that Φ can not take its positive maximum over $\overline{\Omega} \times \overline{\Omega}$ at $(x_0 . y_0) \in \overline{\Omega} \times \Gamma$, we have

$$0 \leqq \max\{-\varepsilon^2 a_{ij}\zeta_{ij} + \varepsilon b_i \zeta_i + u^{\varepsilon} - f,\ u^{\varepsilon} - \psi\} \text{ at } y_0.$$

This implies contradiction for some large μ by the same calculation as in the above. Q.E.D.

<u>Remark</u>(4) We can use the above tequenique to more general singular perturbation problems of obstacle type. More general results will be discussed in the forthcoming paper [9].

§ 2 Singular perturbation with a boundary layer

In this section we study the following special singular perturbations of the obstacle problem:

$$(5)_\varepsilon \qquad \begin{aligned} &\max\{-\varepsilon^2 a_{ij}u^\varepsilon_{ij} + u^\varepsilon,\ u^\varepsilon - \psi\} = 0 \quad \text{a.e. in } \Omega \\ &\qquad\qquad\qquad\qquad u^\varepsilon = 1 \quad \text{on } \Gamma . \end{aligned}$$

We must assume another compatibility condition instead of(A.3):

(A.5) $\qquad \psi \geqq 1 \quad \text{on } \Gamma .$

Theorem 2 Assume hypotheses (A.1), (A.2) and (A.5). For each $\varepsilon > 0$ let u^ε be the unique solution of $(5)_\varepsilon$. Then, we find that there is a positive constant C and a Lipschitz function w(x) such that for all $x \in \overline{\Omega}$

(6) $\qquad -C\varepsilon \leqq u^\varepsilon(x) - u^0(x) \leqq \exp\{-w(x)+o(1)\}/\varepsilon .$

Remark(4) Roughly speaking, w(x) is a distance from the boundary; $w = 0$ on Γ and $w > 0$ in Ω.

Before the proof of Theorem 2 we state the following lemma:

Lemma (L.C.Evans and H.Ishii [4]) Assume hypotheses (A.1) and (A.2). For each $\varepsilon > 0$ let v^ε be the unique solution of the linear equation:

$$(7)_\varepsilon \quad \begin{cases} -\varepsilon^2 a_{ij} v^\varepsilon_{ij} + v^\varepsilon = 0 & \text{in } \Omega, \\ v^\varepsilon = 1 & \text{on } \Gamma \end{cases}$$

Then, we find

$$v^\varepsilon(x) = \exp\{-w(x)+o(1)\}/\varepsilon \quad \text{in } \overline{\Omega},$$

where w is the same function in Remark(4).

Remark(5) We also refer [6] and [8] for the details.

(6) In case that $\min\{\psi(x) \mid x \in \overline{\Omega}\} > 0$ we note that $u^\varepsilon = v^\varepsilon$ on $\overline{\Omega}$ for small ε.

Proof of Theorem 2 In order to obtain $-C\varepsilon \leqq u^\varepsilon - u^0$, we can apply the same technique as in the proof of Theorem 1. So we leave the proof to the reader.

Now let us show the upper estimates.

At first we shall prove $u^\varepsilon - u^0 \leqq \exp\{-w+o(1)\}/\varepsilon$. Let v^ε be the unique solution of $(7)_\varepsilon$. Assume that $u^\varepsilon - u^0 - v^\varepsilon$ attains its positive maximum over $\overline{\Omega}$ at x_0. We can assume that x_0 does not belong to the set $\Omega_0 = \{x \in \overline{\Omega} \mid \psi(x) \leqq 0\}$. Indeed, remarking that $u^\varepsilon - u^0 = u^\varepsilon - \psi \leqq 0$ on Ω_0 and $v^\varepsilon \geqq 0$ on $\overline{\Omega}$, we have $u^\varepsilon - u^0 - v^\varepsilon \leqq 0$ on Ω_0.

Hence, for each $\varepsilon > 0$ there is a neighborhood $N_\varepsilon \subset \Omega$ of x_0

such that $\psi > 0$ in N_ε. Since $u^0 = 0$ in N_ε, $u^\varepsilon - v^\varepsilon$ attains its positive maximum over N_ε at x_0. Bony's maximum principle ([2]) implies that

$$0 \leqq \lim_{x \in N_\varepsilon, x \to x_0} \operatorname{ess\,sup}[-\varepsilon^2 a_{ij}(x)\{u^\varepsilon_{ij}(x) - v^\varepsilon_{ij}(x)\}]$$

Remembering that $-\varepsilon^2 a_{ij} u^\varepsilon_{ij} + u^\varepsilon \leqq 0$ a.e. in Ω, we get contradiction.

Hence, we have $u^\varepsilon - u^0 \leqq v^\varepsilon$ in $\overline{\Omega}$. Combining this with Lemma, we obtain the estimates (6). Q.E.D.

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